ON GEOMETRIC BOTT-CHERN FORMALITY AND DEFORMATIONS

NICOLETTA TARDINI AND ADRIANO TOMASSINI

Abstract. A notion of geometric formality in the context of Bott-Chern and Aeppli cohomologies on a complex manifold is discussed. In particular, by using Aeppli-Bott-Chern-Massey triple products, it is proved that geometric Aeppli-Bott-Chern formality is not stable under small deformations of the complex structure.

Introduction

On a complex manifold one can consider two different kinds of invariants: the topological ones of the underline manifold and the complex ones. Among the first ones a fundamental role is played by de Rham cohomology, among the second ones we recall the Dolbeault, Bott-Chern and Aeppli cohomologies; where, Bott-Chern and Aeppli cohomologies of a complex manifold X are, respectively, defined as

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\mathrm{Ker}\partial \cap \mathrm{Ker}\overline{\partial}}{\mathrm{Im}\partial\overline{\partial}}, \qquad H^{\bullet,\bullet}_{A}(X) := \frac{\mathrm{Ker}\partial\overline{\partial}}{\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}}.$$

Since all the cohomologies just mentioned coincide on a compact Kähler manifold, more precisely $\partial\overline{\partial}$-lemma holds on X (this is in particular true on a Kähler manifold) if and only if the maps induced by identity in the diagram

$$\begin{tikzcd}
& H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{\bullet,\bullet}_{\partial}(X) \arrow[dr] & H^{\bullet}_{dR}(X,\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\overline{\partial}}(X) \arrow[dl] \\
& H^{\bullet,\bullet}_{A}(X) &
\end{tikzcd} .$$

are all isomorphisms, then Bott-Chern and Aeppli cohomologies could provide more informations on the complex structure when X does not admit a Kähler metric.
The theory of formality, developed by Sullivan, concerns with differential-graded-algebras, namely graded algebras endowed with a derivation with square equal to 0. An immediate example is given by the space of differential (resp. complex) forms on a differentiable

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(resp. complex) manifold together with the exterior derivative. It is proven in [6] that compact complex manifolds satisfying $\partial\overline{\partial}$-lemma are formal in the sense of Sullivan.
On the other side, on a complex manifold X the double complex of bigraded forms $\left(\Lambda^{\bullet,\bullet}X, \partial, \overline{\partial}\right)$ is naturally defined; then, one could ask whether a notion of formality could be defined in case of bidifferential-bigraded-algebras. In this context Neisendorfer and Taylor developed a formality theory for the Dolbeault complex on complex manifolds (see [10]). In particular, we are interested in a formality notion for Bott-Chern-cohomology.
Inspired by Kotschick [8], D. Angella and the second author in [3], define a compact complex manifold X being geometrically-H_{BC}-formal if there exists a Hermitian metric g on X such that the space of Δ_{BC}-harmonic forms (in the sense of Schweitzer [13]) has a structure of algebra. Moreover, an obstruction to the existence of such a metric on X is provided by *Aeppli-Bott-Chern-Massey triple products* (see Theorem 2.2).
In this note we are interested in studying the relationship of this new notion with the complex structure, in particular we discuss the behaviour of geometric-H_{BC}-formality under small deformations of the complex structure (see [14] for similar results for Dolbeault formality).
Indeed, considering compact complex surfaces diffeomorphic to solvmanifolds the property considered is open, however, more in general, we prove the following

Theorem 1 (see Theorem 3.1 and Corollary 3.2). *The property of geometric-H_{BC}-formality is not stable under small deformations of the complex structure.*

A key tool in the proof of Theorem 1 is Theorem 2.2. First of all we construct a complex curve J_t of complex structures on $X = \mathbb{S}^3 \times \mathbb{S}^3$ such that J_0 is the the geometrically-H_{BC}-formal Calabi-Eckmann complex structure on X; then, by computing the Bott-Chern cohomology of $X_t = (\mathbb{S}^3 \times \mathbb{S}^3, J_t)$ for small t, we exhibit a non-trivial Aeppli-Bott-Chern Massey triple product on X_t, for $t \neq 0$. Furthermore, we show that the non holomorphically parallelizable Nakamura manifold has no geometrically H_{BC}-formal metric (see Example 2.3).

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1. Bott-Chern cohomology and Aeppli-Bott-Chern geometrical formality

Let X be a compact complex manifold of complex dimension n. We will denote by $A^{p,q}(X)$ the space of complex (p,q)-forms on X. The *Bott-Chern* and *Aeppli cohomology groups* of X are defined respectively as (see [1] and [4])

$$H^{\bullet,\bullet}_{BC}(X) = \frac{\operatorname{Ker}\partial \cap \operatorname{Ker}\overline{\partial}}{\operatorname{Im}\partial\overline{\partial}}, \quad H^{\bullet,\bullet}_{A}(X) = \frac{\operatorname{Ker}\partial\overline{\partial}}{\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}}.$$

Let g be a Hermitian metric on X and $* : A^{p,q}(X) \to A^{n-p,n-q}(X)$ be the complex Hodge operator associated with g. Let $\tilde{\Delta}_{BC}$ and $\tilde{\Delta}_A$ be the 4-th order elliptic self-adjoint differential operators defined respectively as

$$\tilde{\Delta}^g_{BC} := (\partial\overline{\partial})(\partial\overline{\partial})^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial$$

and

$$\tilde{\Delta}^g_{A} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + (\partial\overline{\partial})(\partial\overline{\partial})^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^*.$$

Then, accordingly to [13], it turns out that $H^{\bullet,\bullet}_{BC}(X) \simeq \mathrm{Ker}\,\tilde{\Delta}^g_{BC}$ and $H^{\bullet,\bullet}_{A}(X) \simeq \mathrm{Ker}\,\tilde{\Delta}^g_{A}$, so that $H^{\bullet,\bullet}_{BC}(X)$ and $H^{\bullet,\bullet}_{A}(X)$ are finite dimensional complex vector spaces. Denoting by α a (p,q)-form on X, note that

$$\alpha \in \mathrm{Ker}\,\tilde{\Delta}^g_{BC} \quad\iff\quad \begin{cases} \partial\alpha = 0\,, \\ \overline{\partial}\alpha = 0\,, \\ \partial\overline{\partial} * \alpha = 0\,, \end{cases} \quad\iff\quad *\alpha \in \mathrm{Ker}\,\tilde{\Delta}^g_{A}\,.$$

Therefore, $*$ induces an isomorphism between $H^{p,q}_{BC}(X)$ and $H^{n-p,n-q}_{A}(X)$. Furthermore, the wedge product induces a structure of algebra on $\bigoplus_{p,q} H^{p,q}_{BC}(X)$ and a structure of $\bigoplus_{p,q} H^{p,q}_{BC}(X)$-module on $\bigoplus_{p,q} H^{p,q}_{A}(X)$ (see [13, Lemme 2.5]).

Since in general the wedge product of harmonic forms may be not a harmonic form, the following definition makes sense (see [8] for Riemannian metrics)

Definition 1.1. *A Hermitian metric g on X is said to be* geometrically-H_{BC}-formal *if* $\mathrm{Ker}\,\tilde{\Delta}^g_{BC}$ *is an algebra. Similarly, a compact complex manifold X is said to be* geometrically -H_{BC}-formal *if there exists a geometrically H_{BC}-formal Hermitian metric on X.*

2. Aeppli-Bott-Chern-Massey triple products

Let X be a compact complex manifold and denote by $(A^{\bullet,\bullet}(X), \partial, \overline{\partial})$ the bi-differential bi-graded algebra of (p,q)-forms on X. As we have already noted in section 1, on a compact complex manifold X, the Bott-Chern cohomology has a structure of algebra, instead, the Aeppli cohomology has a structure of $H^{\bullet,\bullet}_{BC}(X)$-module. This motivates the following (see [3])

Definition 2.1. *Take*

$$\mathfrak{a}_{12} \;=\; [\alpha_{12}] \in H^{p,q}_{BC}(X)\,, \quad \mathfrak{a}_{23} \;=\; [\alpha_{23}] \in H^{r,s}_{BC}(X)\,, \quad \mathfrak{a}_{34} \;=\; [\alpha_{34}] \in H^{u,v}_{BC}(X)\,,$$

such that $\mathfrak{a}_{12} \cup \mathfrak{a}_{23} = 0$ in $H^{p+r,q+s}_{BC}(X)$ and $\mathfrak{a}_{23} \cup \mathfrak{a}_{34} = 0$ in $H^{r+u,s+v}_{BC}(X)$: let

$$(-1)^{p+q}\,\alpha_{12} \wedge \alpha_{23} \;=\; \partial\overline{\partial}\alpha_{13} \qquad \textit{and} \qquad (-1)^{r+s}\,\alpha_{23} \wedge \alpha_{34} \;=\; \partial\overline{\partial}\alpha_{24}\,.$$

The Aeppli-Bott-Chern triple Massey product *is defined as*

$$\mathfrak{a}_{1234} \;:=\; \langle \mathfrak{a}_{12},\, \mathfrak{a}_{23},\, \mathfrak{a}_{34} \rangle_{ABC} := \left[(-1)^{p+q}\,\alpha_{12} \wedge \alpha_{24} - (-1)^{r+s}\,\alpha_{13} \wedge \alpha_{34}\right] \in$$
$$\in \frac{H^{p+r+u-1,q+s+v-1}_{A}(X)}{H^{p,q}_{BC}(X) \cup H^{r+u-1,s+v-1}_{A}(X) + H^{p+r-1,q+s-1}_{A}(X) \cup H^{u,v}_{BC}(X)}\,.$$

Similarly to the real case, Aeppli-Bott-Chern Massey triple products provide an obstruction to geometric-H_{BC}-formality (see also [3]).

Theorem 2.2. *Let X be a compact complex manifold. If X is geometrically-H_{BC}-formal then the Aeppli-Bott-Chern Massey triple products are trivial.*

Proof. Fix a Hermitian metric g on X such that $\mathrm{Ker}\Delta^g_{BC}$ has a structure of algebra. Take

$$\mathfrak{a}_{12} \;=\; [\alpha_{12}] \in H^{p,q}_{BC}(X)\,, \quad \mathfrak{a}_{23} \;=\; [\alpha_{23}] \in H^{r,s}_{BC}(X)\,, \quad \mathfrak{a}_{34} \;=\; [\alpha_{34}] \in H^{u,v}_{BC}(X)\,,$$

such that $\mathfrak{a}_{12} \cup \mathfrak{a}_{23} = 0$ in $H^{p+r,q+s}_{BC}(X)$ and $\mathfrak{a}_{23} \cup \mathfrak{a}_{34} = 0$ in $H^{r+u,s+v}_{BC}(X)$, with $\alpha_{12}, \alpha_{23}, \alpha_{34}$ harmonic representatives in the respective classes. Then $\alpha_{12} \wedge \alpha_{23}$ and $\alpha_{23} \wedge \alpha_{34}$ are

harmonic forms with respect to the Laplacian $\tilde{\Delta}^g_{BC}$. Hence, with the notation introduced, $\alpha_{13} = 0$ and $\alpha_{24} = 0$. Therefore, by definition, $\langle \mathfrak{a}_{12},\, \mathfrak{a}_{23},\, \mathfrak{a}_{34}\rangle_{ABC} = 0$. □

Example 2.3. *Let $G = \mathbb{C} \ltimes_\varphi \mathbb{C}^2$, where $\varphi : \mathbb{C} \to \mathrm{GL}(2,\mathbb{C})$ is defined as*

$$\varphi(x+iy) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}.$$

Then for some $a \in \mathbb{R}$ the matrix $\begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}$ is conjugated to a matrix in $\mathrm{SL}(2,\mathbb{Z})$. Then $\Gamma := (a\mathbb{Z} + i2m\pi\mathbb{Z}) \ltimes_\varphi \Gamma''$, with Γ'' lattice in $\mathbb{C}^2$, is a lattice in G (see [2] *and* [9]*). Denoting with (z_1, z_2, z_3) global coordinates on G, the following forms*

$$\psi^1 = dz_1\,, \qquad \psi^2 = e^{-\frac{1}{2}(z_1+\bar{z}_1)}dz_2\,, \qquad \psi^3 = e^{\frac{1}{2}(z_1+\bar{z}_1)}dz_3$$

are Γ-invariant. A direct computation shows that

$$\begin{aligned} &\partial\psi^1 = 0 \quad \partial\psi^2 = -\tfrac{1}{2}\psi^{12} \quad \partial\psi^3 = \tfrac{1}{2}\psi^{13} \\ &\bar{\partial}\psi^1 = 0 \quad \bar{\partial}\psi^2 = \tfrac{1}{2}\psi^{2\bar{1}} \quad\; \bar{\partial}\psi^3 = -\tfrac{1}{2}\psi^{3\bar{1}}\,, \end{aligned}$$

where $\psi^{A\bar{B}} = \psi^A \wedge \bar{\psi}^B$ and so on. Therefore $\{\psi^1, \psi^2, \psi^3\}$ give rise to complex $(1,0)$-forms on the compact manifold $X = \Gamma\backslash G$. We will show that X is not geometrically-H_{BC}-formal. Let

$$\mathfrak{a}_{12} = [e^{\frac{1}{2}(z_1-\bar{z}_1)}\psi^{1\bar{3}}] \in H^{1,1}_{BC}(X)\,, \qquad \mathfrak{a}_{23} = [e^{\frac{1}{2}(z_1-\bar{z}_1)}\psi^{\overline{12}}] \in H^{0,2}_{BC}(X)\,,$$

$$\mathfrak{a}_{34} = [e^{-\frac{1}{2}(z_1-\bar{z}_1)}\psi^{\bar{1}3}] \in H^{1,1}_{BC}(X)\,.$$

Then,

$$e^{z_1-\bar{z}_1}\psi^{1\overline{123}} = \partial\bar{\partial}(-e^{z_1-\bar{z}_1}\psi^{\overline{23}})\,.$$

Therefore,

$$\langle \mathfrak{a}_{12},\, \mathfrak{a}_{23},\, \mathfrak{a}_{34}\rangle_{ABC} = \left[e^{\frac{1}{2}(z_1-\bar{z}_1)}\psi^{\overline{123}3}\right] \in \frac{H^{1,3}_A(X)}{H^{1,1}_{BC}(X) \cup H^{0,2}_A(X)}\,.$$

According to the cohomology computations in [2, table 4]*, it follows that $e^{\frac{1}{2}(z_1-\bar{z}_1)}\psi^{\overline{123}3} \in \operatorname{Ker}\tilde{\Delta}^g_A$. Furthermore, a direct computation shows that $[e^{\frac{1}{2}(z_1-\bar{z}_1)}\psi^{\overline{123}3}] \notin H^{1,1}_{BC}(X) \cup H^{0,2}_A(X)$, so that $\langle \mathfrak{a}_{12},\, \mathfrak{a}_{23},\, \mathfrak{a}_{34}\rangle_{ABC}$ is a non-trivial Aeppli-Bott-Chern Massey product.*

3. Instability of Bott-Chern geometrical formality

In this section, starting with a geometrical-H_{BC}-formal compact complex manifold, we will construct a complex deformation which is no more geometrically-H_{BC}-formal.

Let $X = \mathbb{S}^3 \times \mathbb{S}^3$ and $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ be the Lie group of special unitary 2×2 matrices and denote by $\mathfrak{su}(2)$ the Lie algebra of $\mathrm{SU}(2)$. Denote by $\{e_1, e_2, e_3\}$, $\{f_1, f_2, f_3\}$ a basis of the first copy of $\mathfrak{su}(2)$, respectively of the second copy of $\mathfrak{su}(2)$ and by $\{e^1, e^2, e^3\}$, $\{f^1, f^2, f^3\}$ the corresponding dual co-frames. Then we have the following commutation rules:

$$[e_1, e_2] = 2e_3\,, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1\,,$$

and the corresponding Cartan structure equations

$$
\left\{
\begin{array}{rcl}
de^1 & = & -2\, e^2 \wedge e^3 \\
de^2 & = & 2\, e^1 \wedge e^3 \\
de^3 & = & -2\, e^1 \wedge e^2 \\
df^1 & = & -2\, f^2 \wedge f^3 \\
df^2 & = & 2\, f^1 \wedge f^3 \\
df^3 & = & -2\, f^1 \wedge f^2
\end{array}
\right. . \tag{1}
$$

Define a complex structure J on X by setting

$$
Je_1 = e_2, \quad Jf_1 = f_2, \quad Je_3 = f_3 .
$$

Note that J is a Calabi-Eckmann structure or its conjugate on $\mathbb{S}^3 \times \mathbb{S}^3$ (see [5] and [11]).

We have the following

Theorem 3.1. *Let $X = \mathbb{S}^3 \times \mathbb{S}^3$ be endowed with the complex structure J. Then X is geometrically-H_{BC}-formal and there exists a small deformation $\{X_t\}$ of X such that X_t is not geometrically-H_{BC}-formal for $t \neq 0$.*

Proof. For the sake of the completeness we will recall the proof of geometric H_{BC}-formality of X (see [3]). According to the previous notation, a complex co-frame of $(1,0)$-forms for J is given by

$$
\left\{
\begin{array}{rcl}
\varphi^1 & := & e^1 + i\, e^2 \\
\varphi^2 & := & f^1 + i\, f^2 \\
\varphi^3 & := & e^3 + i\, f^3
\end{array}
\right. .
$$

Therefore the complex structure equations are given by

$$
\left\{
\begin{array}{rcl}
d\varphi^1 & = & i\varphi^1 \wedge \varphi^3 + i\varphi^1 \wedge \overline{\varphi}^3 \\
d\varphi^2 & = & \varphi^2 \wedge \varphi^3 - \varphi^2 \wedge \overline{\varphi}^3 \\
d\varphi^3 & = & -i\varphi^1 \wedge \overline{\varphi}^1 + \varphi^2 \wedge \overline{\varphi}^2
\end{array}
\right. ,
$$

in particular,

$$
\left\{
\begin{array}{rcl}
\partial\varphi^1 & = & i\varphi^1 \wedge \varphi^3 \\
\partial\varphi^2 & = & \varphi^2 \wedge \varphi^3 \\
\partial\varphi^3 & = & 0
\end{array}
\right.
\qquad \text{and} \qquad
\left\{
\begin{array}{rcl}
\overline{\partial}\varphi^1 & = & i\varphi^1 \wedge \overline{\varphi}^3 \\
\overline{\partial}\varphi^2 & = & -\varphi^2 \wedge \overline{\varphi}^3 \\
\overline{\partial}\varphi^3 & = & -i\varphi^1 \wedge \overline{\varphi}^1 + \varphi^2 \wedge \overline{\varphi}^2
\end{array}
\right. .
$$

Now fix the Hermitian metric whose associated fundamental form is

$$
\omega \;:=\; \frac{i}{2} \sum_{j=1}^{3} \varphi^j \wedge \overline{\varphi}^j .
$$

As a matter of notation, from now on we shorten, e.g., $\varphi^{1\overline{1}} := \varphi^1 \wedge \overline{\varphi}^1$.
As regards the Bott-Chern cohomology, thanks to [2, Theorem 1.3] we have that the sub-complex

$$
\iota \colon \wedge \left\langle \varphi^1, \varphi^2, \varphi^3, \bar{\varphi}^1, \bar{\varphi}^2, \bar{\varphi}^3 \right\rangle \hookrightarrow A^{\bullet,\bullet}(X)
$$

is such that $H_{BC}(\iota)$ is an isomorphism, hence, by explicit computations, we get

$$
\begin{array}{rcl}
H^{0,0}_{BC}(X) & = & \mathbb{C}\left\langle [1] \right\rangle, \\
H^{1,1}_{BC}(X) & = & \mathbb{C}\left\langle \left[\varphi^{1\bar{1}}\right], \left[\varphi^{2\bar{2}}\right] \right\rangle, \\
H^{2,1}_{BC}(X) & = & \mathbb{C}\left\langle \left[\varphi^{23\bar{2}} + i\varphi^{13\bar{1}}\right] \right\rangle, \\
H^{1,2}_{BC}(X) & = & \mathbb{C}\left\langle \left[\varphi^{2\overline{23}} - i\varphi^{1\overline{13}}\right] \right\rangle, \\
H^{2,2}_{BC}(X) & = & \mathbb{C}\left\langle \left[\varphi^{12\overline{12}}\right] \right\rangle, \\
H^{3,2}_{BC}(X) & = & \mathbb{C}\left\langle \left[\varphi^{123\overline{12}}\right] \right\rangle, \\
H^{2,3}_{BC}(X) & = & \mathbb{C}\left\langle \left[\varphi^{12\overline{123}}\right] \right\rangle, \\
H^{3,3}_{BC}(X) & = & \mathbb{C}\left\langle \left[\varphi^{123\overline{123}}\right] \right\rangle.
\end{array}
$$

The other Bott-Chern cohomology groups are trivial.
Notice that the Hermitian metric g associated to ω is geometrically-H_{BC}-formal, hence $\mathbb{S}^3 \times \mathbb{S}^3$ is geometrically-H_{BC}-formal. Now our purpose is to prove that geometrical-H_{BC}-formality is not stable under small deformations of the complex structure. In order to get this result, let J_t be the almost complex structure on X defined as

$$
\left\{
\begin{array}{rcl}
\varphi^1_t & := & \varphi^1 \\
\varphi^2_t & := & \varphi^2 \\
\varphi^3_t & := & \varphi^3 - t\overline{\varphi}^3
\end{array}
\right.
$$

then

$$
\left\{
\begin{array}{rcl}
d\varphi^1_t & = & \frac{i(\bar{t}+1)}{1-|t|^2}\varphi^{13}_t + \frac{i(t+1)}{1-|t|^2}\varphi^{1\bar{3}}_t \\
d\varphi^2_t & = & \frac{1-\bar{t}}{1-|t|^2}\varphi^{23}_t + \frac{t-1}{1-|t|^2}\varphi^{2\bar{3}}_t \\
d\varphi^3_t & = & (t-i)\varphi^{1\bar{1}}_t + (t+1)\varphi^{2\bar{2}}_t
\end{array}
\right. .
$$

and consequently J_t is integrable. Set $X_t = (X, J_t)$ and g_t the Hermitian metric whose fundamental form is $\omega_t = \frac{i}{2}\sum \varphi^j_t \wedge \overline{\varphi}^j_t$. By applying again [2, Theorem 1.3], we compute the Bott-Chern cohomology of X_t;

if $|t|^2 + \Re\mathfrak{e}\, t - \Im\mathfrak{m}\, t \neq 0$ we get

$$\begin{aligned}
H^{0,0}_{BC}(X_t) &= \mathbb{C}\left\langle [1] \right\rangle , \\
H^{1,1}_{BC}(X_t) &= \mathbb{C}\left\langle \left[\varphi^{1\bar{1}}_t\right], \left[\varphi^{2\bar{2}}_t\right] \right\rangle , \\
H^{2,1}_{BC}(X_t) &= \mathbb{C}\left\langle \left[\varphi^{23\bar{2}}_t + \frac{i-t}{t+1}\varphi^{13\bar{1}}_t\right] \right\rangle , \\
H^{1,2}_{BC}(X_t) &= \mathbb{C}\left\langle \left[\varphi^{2\bar{2}\bar{3}}_t - \frac{i+\bar{t}}{t+1}\varphi^{1\bar{1}\bar{3}}_t\right] \right\rangle , \\
H^{3,2}_{BC}(X_t) &= \mathbb{C}\left\langle \left[\varphi^{123\bar{1}\bar{2}}_t\right] \right\rangle , \\
H^{2,3}_{BC}(X_t) &= \mathbb{C}\left\langle \left[\varphi^{12\bar{1}\bar{2}\bar{3}}_t\right] \right\rangle , \\
H^{3,3}_{BC}(X_t) &= \mathbb{C}\left\langle \left[\varphi^{123\bar{1}\bar{2}\bar{3}}_t\right] \right\rangle ,
\end{aligned}$$

where the other groups are trivial, in particular $H^{2,2}_{BC}(X, J_t)$ vanishes.
Now we will show that there are no geometrical-H_{BC}-formal Hermitian metric on X_t. To this purpose we are going to exhibit a non-trivial ABC-Massey triple product.
Setting

$$\begin{aligned}
\mathfrak{a}_{12} &= \left[\varphi^{1\bar{1}}_t\right] \in H^{1,1}_{BC}(X_t) , \\
\mathfrak{a}_{23} &= \left[\varphi^{2\bar{2}}_t\right] \in H^{1,1}_{BC}(X_t) , \\
\mathfrak{a}_{34} &= \left[\varphi^{2\bar{2}}_t\right] \in H^{1,1}_{BC}(X_t) .
\end{aligned}$$

we get $\mathfrak{a}_{12} \cup \mathfrak{a}_{23} = \mathfrak{a}_{23} \cup \mathfrak{a}_{34} = 0$. Indeed

$$\partial\bar{\partial}\varphi^{3\bar{3}}_t = \left[(t-i)(\bar{t}+1) + (t+1)(\bar{t}+i)\right]\varphi^{12\bar{1}\bar{2}}_t =: A_t\varphi^{12\bar{1}\bar{2}}_t,$$

so, in the hypothesis that $|t|^2 + \Re\mathfrak{e}\, t - \Im\mathfrak{m}\, t \neq 0$, we can take as representatives

$$\alpha_{13} = -\frac{1}{A_t}\varphi^{3\bar{3}}_t, \qquad \alpha_{24} = 0.$$

Thus the corresponding ABC-Massey product is

$$\langle \mathfrak{a}_{12},\, \mathfrak{a}_{23},\, \mathfrak{a}_{34} \rangle_{ABC} \quad = \quad \left[-\frac{1}{A_t}\varphi^{23\bar{2}\bar{3}}_t\right] \in \frac{H^{2,2}_A}{H^{1,1}_{BC} \smile H^{1,1}_A + H^{1,1}_A \smile H^{1,1}_{BC}}(X_t) \ .$$

It is easy to check that this ABC-Massey triple product is not zero, in fact $\left[-\frac{1}{A_t}\varphi^{23\bar{2}\bar{3}}_t\right] \neq 0$ in $H^{2,2}_A(X_t)$ since $-\frac{1}{A_t}\varphi^{23\bar{2}\bar{3}}_t$ is $\tilde{\Delta}^{g_t}_A$-harmonic in X_t. Furthermore $H^{1,1}_A(X_t) = \{0\}$, since $H^{2,2}_{BC}(X_t) = \{0\}$.
This conclusion the proof. □

As a consequence, we obtain the following

Corollary 3.2. *The property of geometric-H_{BC}-formality is not stable under small deformations of the complex structure.*

Remark 3.3. *Recall that a Hermitian metric g on a complex manifold X of dimension n is said to be* strong Kähler with torsion *(shortly* SKT*), respectively* Gauduchon*, if its fundamental form ω satisfies $\partial\overline{\partial}\omega = 0$, respectively $\partial\overline{\partial}\omega^{n-1} = 0$.*
A direct computation shows that the Hermitian metric $\omega := \frac{i}{2}\sum_{j=1}^{3}\varphi^j \wedge \overline{\varphi}^j$ defined on $\mathbb{S}^3 \times \mathbb{S}^3$ is SKT and Gauduchon. As regard the deformation previously considered, there are two different situations for X_t in a small neighborhood of $t = 0$. If $|t|^2 + \mathfrak{Re}\, t - \mathfrak{Im}\, t \neq 0$ the Hermitian metric $\omega_t := \frac{i}{2}\sum \varphi_t^j \wedge \overline{\varphi}_t^j$ is not SKT, and, more precisely, X_t does not admit such a metric.

Indeed, let

$$\partial\overline{\partial}\varphi_t^{3\bar{3}} = -2\left(|t|^2 + \mathfrak{Re}\, t - \mathfrak{Im}\, t\right)\varphi_t^{1\bar{1}2\bar{2}} =: U_t;$$

then, for $|t|^2 + \mathfrak{Re}\, t - \mathfrak{Im}\, t > 0$, the $(2,2)$-form U_t gives rise to a $\partial\overline{\partial}$-exact $(1,1)$-positive non-zero current on X_t. Then, in view of the characterisation Theorem of the existence of SKT metrics in terms of currents (see [7] *and* [12]*), it follows that X_t has no SKT metrics for $|t|^2 + \mathfrak{Re}\, t - \mathfrak{Im}\, t > 0$.*

Otherwise, if $|t|^2 + \mathfrak{Re}\, t - \mathfrak{Im}\, t = 0$, a straightforward computation shows that the Hermitian metric ω_t is SKT.

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Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127, Pisa, Italy

E-mail address: `tardini@mail.dm.unipi.it`

Dipartimento di Matematica e Informatica, Università di Parma, Viale Parco Area delle Scienze 53/A, 43124, Parma, Italy

E-mail address: `adriano.tomassini@unipr.it`